

IN WHICH DIMENSIONS IS THE BALL RELATIVELY WORST PACKING?

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ABSTRACT. It was conjectured by Ulam that the ball has the lowest optimal lattice packing density out of all convex, origin-symmetric three-dimensional solids. We affirm a local version of this conjecture: the ball has a lower optimal lattice packing than any body of sufficiently small asphericity in three dimensions. We also show that in dimensions 4, 5, 6, 7, 8, and 24 there are bodies of arbitrarily small asphericity that pack worse than balls.

1. INTRODUCTION

When Martin Gardner’s “New Mathematical Diversions”, collecting some of his *Mathematical Games* columns, was reprinted in 1995, the column on “Packing Spheres” appeared with a postscript, in which Gardner writes “Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove” [3]. In other words, Ulam “suspected that spheres, in their densest packing, allow more empty space than the densest packing of any identical convex solids” [4]. Since too many conjectures already bear the name *Ulam’s conjecture*, it would be appropriate to refer to this one as *Ulam’s packing conjecture*, or perhaps even as *Ulam’s last conjecture*, seeing how it too was published posthumously and how the space in Gardner’s postscript evidently could not fit any motivation for Ulam’s suspicion. For the purposes of this article, we restrict our attention only to solids that are origin-symmetric and only to lattice packings.

The suspicion is not hard to motivate naively by the fact that the sphere is the most symmetric solid and therefore also the least free: in placing a sphere in space there are only three degrees of freedom, compared to five in the case of any other solid of revolution, and six in the case of any other solid. Therefore, it is natural to suspect that if we break the rotational symmetry of the sphere, we introduce more freedom, which could be used to tighten up the packing and bring the packing density above the optimal packing density of spheres.

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The naivety of this motivation is revealed by the fact that while one would expect it to work equally well in the plane, where the circle is the least free domain, the circle can actually be packed more densely than some other domains. A packing of circles can cover $0.9068\dots$ of the plane, while a packing of regular octagons cannot cover more than $0.9061\dots$. In 1934, Reinhardt conjectured that a smoothed octagon he constructed has the lowest optimal packing density of any origin-symmetric convex domain [10]. A packing of smoothed octagons can cover at most $0.9024\dots$ of the plane. In fact, the circle is not even a local minimum of the optimal packing density. There are origin-symmetric shapes arbitrarily circular (the outradius and inradius both being arbitrarily close to 1) that pack worse than circles [7]. Ulam's conjecture implies that this would not be the case in three dimensions. The question then is in which dimensions the ball is a local minimum of the optimal packing density. In this article, we answer this question for all dimensions in which the highest lattice packing density for spheres is known.

In Section 2 we provide some preliminaries about convex bodies and admissible lattice. In Section 3 we introduce the notions and perfection and eutaxy and their relations to the question at hand. We observe that these relations are strong enough to determine that the ball is not relatively worst packing in dimensions 4, 5, 6, 7, 8, and 24. The three dimensional case is treated in Section 4, where the ball is proved to be relatively worst packing.

2. CONVEX BODIES AND ADMISSIBLE LATTICES

An n -dimensional *convex body* is a convex, compact subset of \mathbb{R}^n with a nonempty interior. A body K is *symmetric about the origin* (or origin-symmetric) if $-K = K$. In this article we discuss only such bodies, and we will implicitly assume that every body mentioned is symmetric about the origin. We denote by B^n the Euclidean unit ball of \mathbb{R}^n . The space of origin-symmetric convex bodies \mathcal{K}_0^n in \mathbb{R}^n is a metric space equipped with the Hausdorff metric $\delta_H(K, K') = \min\{\varepsilon : K \subseteq K' + \varepsilon B^n, K' \subseteq K + \varepsilon B^n\}$. The set of bodies K satisfying $aB^n \subseteq K \subseteq bB^n$ for $b > a > 0$ is compact [5].

Let $S^{n-1} = \partial B^n$ be the unit sphere. The *support height* of an n -dimensional body in the direction $\mathbf{x} \in S^{n-1}$ is given by $h_K(\mathbf{x}) = \max_{\mathbf{y} \in K} \langle \mathbf{x}, \mathbf{y} \rangle$. The half-space $H_K(\mathbf{x}) = \{\mathbf{y} : \langle \mathbf{x}, \mathbf{y} \rangle \leq h_K(\mathbf{x})\}$ contains K , and a body is uniquely determined by its support height function since $K = \bigcap_{\mathbf{x} \in S^{n-1}} H_K(\mathbf{x})$. Similarly, the *radial distance* of a body in the direction $\mathbf{x} \in S^{n-1}$ is given by $r_K(\mathbf{x}) = \max_{\lambda \mathbf{x} \in K} \lambda$. Again, a body

is uniquely determined by its radial distance function. Under non-singular linear transformations $K \mapsto TK$, the radial distance function transforms according to the formula

$$r_{TK}(\mathbf{x}) = \frac{r_K\left(\frac{T^{-1}\mathbf{x}}{\|T^{-1}\mathbf{x}\|}\right)}{\|T^{-1}\mathbf{x}\|}.$$

For origin-symmetric bodies, both the support height and radial distance are even functions.

An n -dimensional *lattice* is the image of the integer lattice \mathbb{Z}^n under some non-singular linear map T . The determinant $d(\Lambda)$ of a lattice $\Lambda = T\mathbb{Z}^n$ is the volume of the image of the unit cube under T and is given by $d(\Lambda) = |\det T|$. The space \mathcal{L}^n of n -dimensional lattices can be equipped with the metric $\delta(\Lambda, \Lambda') = \min\{\|T - T'\| : \Lambda = T\mathbb{Z}^n, \Lambda' = T'\mathbb{Z}^n\}$, where $\|\cdot\|$ is the Hilbert-Schmidt norm. A lattice is called *admissible* for a body K if $\Lambda \cap \text{int } K = \{0\}$. A lattice Λ is admissible for K if and only if $\{K + 2\mathbf{l} : \mathbf{l} \in \Lambda\}$ is a packing, i.e., $K + 2\mathbf{l}$ and $K + 2\mathbf{l}'$, have disjoint interiors for $\mathbf{l}, \mathbf{l}' \in \Lambda$, $\mathbf{l} \neq \mathbf{l}'$. The fraction of space covered by this packing is $2^{-n} \text{vol } K / d(\Lambda)$. The set of lattices admissible for some body K and of determinant at most some value is compact [5].

The *critical determinant* d_K is the minimum, necessarily attained due to compactness, of all determinants of lattices admissible for K . A lattice attaining this minimum is called a *critical lattice* of K . If an admissible lattice locally minimizes the determinant amongst admissible lattices, it is called an *extreme lattice* of K . Clearly, if $K' \subseteq K$, then $d_{K'} \leq d_K$. If this inequality is strict whenever K' is a proper subset of K , we say that K is an *irreducible* body. The optimal packing fraction for K is $\varphi(K) = 2^{-n} \text{vol } K / d_K$. Note that $\varphi(TK) = \varphi(K)$ for any nonsingular linear transformation T . Therefore, we may define φ as a function over the space of linear classes of n -dimensional bodies, equipped with the Banach-Mazur distance $\delta_{BM}([K], [L]) = \min\{t : L' \subseteq K' \subseteq e^t L', K' \in [K], L' \in [L]\}$. Since this space is compact, there must be a body K with the lowest possible optimal packing fraction amongst all n -dimensional bodies. We call this an *absolutely worst packing* body. If a body belongs to a class which is a local minimum of φ in this space, we say it is *relatively worst packing*. A relatively worst packing body is necessarily irreducible, but the converse is not necessarily true.

Reinhardt's smoothed octagon is known to be relatively worst packing and is conjectured to be absolutely worst packing in two dimensions [9]. The unit disk in two dimensions is irreducible, but is not relatively worst packing [7]. Below we show that the unit ball is relatively worst

packing for $n = 3$, irreducible but not relatively worst packing for $n = 4$ and 5, and reducible for $n = 6, 7, 8$, and 24.

3. PERFECTION AND EUTAXY

Let $S \subset S^{n-1}$ be a finite, origin-symmetric set of unit vectors (that is $\mathbf{x} \in S$ if and only if $-\mathbf{x} \in S$). Every vector of S defines a projection map $P_{\mathbf{x}}(\cdot) = \langle \cdot, \mathbf{x} \rangle \mathbf{x}$. If these projection maps span the space Sym^n of symmetric linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$, we say that S is a *perfect* configuration. The space Sym^n , equipped with inner product $\langle Q, Q' \rangle = \text{trace}(QQ')$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$. Therefore, a perfect configuration must have at least $n(n+1)$ vectors [8].

If there are numbers $v_{\mathbf{x}}$, $\mathbf{x} \in S$, such that $v_{\mathbf{x}} = v_{-\mathbf{x}}$ and $\text{Id} = \sum_{\mathbf{x} \in S} v_{\mathbf{x}} P_{\mathbf{x}}$, then we say that S is a *weakly eutactic* configuration and we call the coefficients $v_{\mathbf{x}}$ eutaxy coefficients [8]. A configuration is *eutactic* (resp. *semi-eutactic*) if it has positive (resp. non-negative) eutaxy coefficients. Let S be perfect and eutactic, then we say that S is *redundantly semi-eutactic* (resp. *redundantly perfect eutactic*) if the configuration $S \setminus \{\mathbf{x}, -\mathbf{x}\}$ is semi-eutactic (resp. perfect and eutactic) for all $\mathbf{x} \in S$. If S is perfect and eutactic and has the minimal number of vectors $n(n+1)$, we say the configuration is *critically eutactic*. The eutaxy coefficients of a critically eutactic configuration are unique and the configuration is not redundantly semi-eutactic.

The *minimal norm* $m(\Lambda)$ of a lattice Λ is the minimum of $\langle \mathbf{x}, \mathbf{x} \rangle$ over $\mathbf{x} \in \Lambda \setminus \{0\}$, and the set of points $S(\Lambda) \subseteq \Lambda$ that achieve this value are the *minimal vectors* of Λ . We say that a lattice Λ with minimal norm $m(\Lambda) = 1$ is *perfect* (resp. *eutactic*, *semi-eutactic*, and so on), if the configuration of its minimal vectors is perfect (resp. eutactic, semi-eutactic, and so on).

The critical lattices of B^n are known for $n \leq 8$ and $n = 24$ [1, 2]. In all of these cases, they are unique up to rotation. For $n = 2, 3, 4, 5, 6, 7, 8$, the critical lattices of B^n are given by appropriate scalings (such that $m(\Lambda) = 1$) of the root lattices $A_2, D_3, D_4, D_5, E_6, E_7$, and E_8 . For $n = 24$, the Leech lattice Λ_{24} is the unique critical lattice of B^n . When we refer to any of these lattices below, we are referring to the appropriately scaled versions. It is easy to check that A_2 and D_3 are critically eutactic, D_4 and D_5 are redundantly semi-eutactic but not redundantly perfect and eutactic, and E_6, E_7, E_8 , and Λ_{24} are redundantly perfect and eutactic. In all cases, the automorphism group of the lattice acts transitively on its minimal vectors and it is therefore possible to choose the eutaxy coefficients so that they are all equal (this is the unique choice when the lattice is critically eutactic).

We establish a relation between eutaxy properties and the existence or non-existence of linear maps which satisfy certain conditions. For this purpose we recall the fundamental theorem of linear algebra, which in our case implies that there exists a symmetric linear map Q such that

$$\langle P_{\mathbf{x}}, Q \rangle = a_{\mathbf{x}}, \quad \langle \text{Id}, Q \rangle = -a_0,$$

if and only if $\sum_{\mathbf{x} \in S} a_{\mathbf{x}} b_{\mathbf{x}} + a_0 b_0 = 0$ whenever the coefficients $b_0, (b_{\mathbf{x}})_{\mathbf{x} \in S}$ are a solution to the equation

$$(1) \quad \sum_{\mathbf{x} \in S} b_{\mathbf{x}} P_{\mathbf{x}} - b_0 \text{Id} = 0,$$

In other words, the space of possible a 's is the orthogonal complement of the space of possible b 's. Also recall that a subspace of \mathbb{R}^m contains a positive vector, if and only if its orthogonal complement does not contain a non-zero, non-negative vector. With these facts in mind we prove the following useful theorem:

Theorem 1. (i) *If S is perfect and eutactic, there exists $\varepsilon > 0$ such that whenever T is a linear map, $\|T - \text{Id}\| < \varepsilon$, and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, then $\det T > 1$ or T is orthogonal.*

(ii) *If S is not perfect or not eutactic then for arbitrarily small $\varepsilon > 0$ there exists a linear map T satisfying $\|T - \text{Id}\| < \varepsilon$, $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, and $\det T < 1$.*

(iii) *S is semi-eutactic if and only if there exists $\varepsilon > 0$ and $C > 0$ such that whenever T is a linear map, $\|T - \text{Id}\| < \varepsilon$, and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, then $\det T > 1 - C\|T - \text{Id}\|^2$.*

(iv) *Let S be critically eutactic with eutaxy coefficients $v_{\mathbf{x}}$ and let numbers $\omega_{\mathbf{x}}$ be given for $\mathbf{x} \in S$ such that $\omega_{\mathbf{x}} = \omega_{-\mathbf{x}}$, then there exists a symmetric linear map $Q \in \text{Sym}^n$ satisfying $\langle \mathbf{x}, Q\mathbf{x} \rangle = \omega_{\mathbf{x}}$ and $\text{trace } Q = \omega_0$ if and only if $\omega_0 = \sum_{\mathbf{x} \in S} \omega_{\mathbf{x}} v_{\mathbf{x}}$.*

Proof. (i) By eutaxy, there exist positive coefficients $b_{\mathbf{x}}$ and b_0 satisfying (1). Therefore, every symmetric map Q such that

$$(2) \quad \langle P_{\mathbf{x}}, Q \rangle \geq 0, \text{ for all } \mathbf{x} \in S$$

and $\langle \text{Id}, Q \rangle \leq 0$ must satisfy $\langle P_{\mathbf{x}}, Q \rangle = 0$ for all $\mathbf{x} \in S$, and therefore, by perfection $Q = 0$. Thus, if $Q \neq 0$ satisfies (2) then $\langle Q, \text{Id} \rangle > 0$. In fact, by compactness of the unit sphere in Sym^n , there must be a positive number $C > 0$ such that $\langle Q, \text{Id} \rangle > C\|Q\|$ whenever $Q \neq 0$ satisfies (2). Now let T be a nonorthogonal linear map such that $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$. Then $Q = T^T T - \text{Id} \neq 0$ satisfies (2), and therefore

$\text{trace } Q = \langle Q, \text{Id} \rangle > C\|Q\|$. For each C , there exists ε such that $\det T = \sqrt{\det(\text{Id} + Q)} > 1$ whenever $\|T - \text{Id}\| < \varepsilon$.

- (ii) If S is not eutactic, there is no solution with positive coefficients to (1), and therefore there is a non-zero solution to (2) with $\langle Q, \text{Id} \rangle \leq 0$. Similarly, if S is eutactic but not perfect, any map Q in the complement of the linear span of $P_{\mathbf{x}}$, $\mathbf{x} \in S$, satisfies (2) and $\langle Q, \text{Id} \rangle = 0$. Consider the map $T_\alpha = \sqrt{\text{Id} + \alpha Q}$, where the square root indicates the unique positive-definite square root. Note that $\|T_\alpha \mathbf{x}\| \geq \|\mathbf{x}\|$, T_α is not orthogonal, $\det T_\alpha = \sqrt{\det(1 + \alpha Q)} < (1 + \frac{\alpha}{n} \text{trace } Q)^{n/2} \leq 1$, and $\|T_\alpha - \text{Id}\|$ can be arbitrarily small.
- (iii) Suppose S is semi-eutactic, then there exist a non-zero solution to (1) with non-negative coefficients. Therefore, there is no symmetric map Q such that

$$(3) \quad \langle Q, P_{\mathbf{x}} \rangle > 0, \text{ for all } \mathbf{x} \in S$$

and $\langle Q, \text{Id} \rangle < 0$. In fact there must also be no symmetric map Q satisfying (2) and $\langle Q, \text{Id} \rangle < 0$, since $Q' = Q + \frac{1}{2n} \text{Id}$ would satisfy (3) and $\langle Q', \text{Id} \rangle < 0$. Again let T be a linear map such that $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$. Then $Q = T^T T - \text{Id}$ satisfies (2), and therefore $\text{trace } Q \geq 0$. If ε is sufficiently small, then $\det T = \sqrt{\det(\text{Id} + Q)} > 1 - C\|T - \text{Id}\|^2$.

If S is not semi-eutactic, then there is no non-zero solution to (1) with non-negative coefficients and therefore, there exists a symmetric map Q satisfying (3) and $\text{trace } Q < 0$. Again, consider the map $T_\alpha = \sqrt{\text{Id} + \alpha Q}$ and note that $\|T_\alpha \mathbf{x}\| > \|\mathbf{x}\|$, $\det T_\alpha = \sqrt{\det(1 + \alpha Q)} \leq (1 + \frac{\alpha}{n} \text{trace } Q)^{n/2} \leq 1 + \alpha \text{trace } Q$, and $\|T_\alpha - \text{Id}\|$ can be arbitrarily small.

- (iv) Since S is critically eutactic, there is a unique solution up to scaling to (1). Therefore, there is a symmetric map Q satisfying $\langle \mathbf{x}, Q\mathbf{x} \rangle = \langle Q, P_{\mathbf{x}} \rangle \geq \omega_{\mathbf{x}}$ and $\text{trace } Q = \langle Q, \text{Id} \rangle \leq \omega_0$ if and only if $\sum_{\mathbf{x} \in S} v_{\mathbf{x}} \omega_{\mathbf{x}} - \omega_0 = 0$. In fact, the map Q is unique and depends linearly on the variables $\omega_{\mathbf{x}}$.

□

Corollary 1. (Voronoi) Λ is an extreme lattice of B^n if and only if $m(\Lambda) = 1$, and Λ is perfect and eutactic.

Proof. Λ is an extreme lattice of B^n if and only if there exists $\varepsilon > 0$ such that whenever $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq 1$ for all $\mathbf{x} \in S(\Lambda)$ then $d(T\Lambda) > d(\Lambda)$ [8]. □

A consequence of this corollary is that there can be only finitely many (up to rotation) critical lattices for B^n . Therefore, for each $\varepsilon > 0$, there

exists some $\varepsilon' > 0$ so that if Λ is admissible for B^n and $d(\Lambda) < d_{B^n} + \varepsilon'$ then there exists a lattice Λ' , critical for B^n such that $\delta(\Lambda, \Lambda') < \varepsilon$. Also, if Λ is critical for a nearly spherical body K satisfying $(1 - \varepsilon')B^n \subseteq K \subseteq (1 + \varepsilon')B^n$, then again there exists a lattice Λ' , critical for B^n such that $\delta(\Lambda, \Lambda') < \varepsilon$.

Theorem 2. *Let Λ be the unique critical lattice of B^n up to rotations. Then B^n is reducible if and only if Λ is redundantly eutactic.*

Proof. Consider the ε -symmetrically truncated ball $B_\varepsilon = \{\mathbf{x} \in B^n : -1 + \varepsilon \leq \langle \mathbf{x}, \mathbf{p} \rangle \leq 1 - \varepsilon\}$, where $\mathbf{p} \in S^{n-1}$ is some arbitrarily chosen “north pole”. First assume that Λ is not redundantly eutactic. That is, we assume that there exists $\mathbf{x}_0 \in S(\Lambda)$ such that $S(\Lambda) \setminus \{\pm \mathbf{x}_0\}$ is not perfect or not eutactic. We are free to assume that Λ is rotated so that $\mathbf{x}_0 = \mathbf{p}$. Then by Theorem 1 (ii), there exists a nonorthogonal linear map T satisfying $\|T\mathbf{x}\| \geq 1$ for all $\mathbf{x} \in S(\Lambda) \setminus \{\pm \mathbf{p}\}$, $\det T < 1$, and $\|T - \text{Id}\|$ is arbitrarily small. In fact, if $\|T - \text{Id}\|$ is small enough, then $T\mathbf{p} \notin B_\varepsilon$ and $T\Lambda$ is admissible for B_ε . Since $d(T\Lambda) < d(\Lambda)$, and since for each proper $K \subset B^n$, there exists $\varepsilon > 0$ such that $K \subset B_\varepsilon \subset B^n$, it follows that B^n is irreducible.

Now suppose that Λ is redundantly eutactic. Let $T\Lambda$ be a critical lattice of B_ε . $\|T - \text{Id}\|$ can be made arbitrarily small by choosing ε sufficiently small and appropriately rotating Λ . If ε is small enough, then because $T\Lambda$ is admissible for B_ε , of the vectors of $TS(\Lambda)$ only one pair $\pm T\mathbf{x}_0$ can be within the interior of the unit ball B^n . Since Λ is redundantly eutactic, the requirement that $\|T\mathbf{x}\| \geq 1$ whenever $\mathbf{x} \in S(\Lambda) \setminus \{\pm \mathbf{x}_0\}$, necessarily implies, when $\|T - \text{Id}\|$ is small enough, that $\det T \geq 1$. Of course, since $d_{B_\varepsilon} \leq d_{B^n}$, we have that $\det T = 1$ and B^n is reducible. \square

Corollary 2. *For $n = 6, 7, 8$, or 24 , the unit ball B^n is reducible.*

Theorem 3. *If Λ is the unique critical lattice of B^n up to rotations and Λ is redundantly semi-eutactic then B^n is not relatively worst packing.*

Proof. Let us call the convex hull of $B_{1/100}$ and $(1 - \varepsilon)B^n$, the ε -shaved ball B'_ε . Let $T\Lambda$ be a critical lattice of B'_ε . Again, if ε is small enough, then of the vectors of $TS(\Lambda)$ only one pair $\pm T\mathbf{x}_0$ can be within the interior of the unit ball B^n . Since Λ is redundantly semi-eutactic, the requirement that $\|T\mathbf{x}\| \geq 1$ whenever $\mathbf{x} \in S(\Lambda) \setminus \{\pm \mathbf{x}_0\}$, necessarily implies, when $\|T - \text{Id}\|$ is small enough, that $\det T > 1 - C\|T - \text{Id}\|^2$. However, note that $\|T - \text{Id}\| < \varepsilon$ and $\text{vol } B'_\varepsilon / \text{vol } B^n < 1 - c\varepsilon$. Therefore, for small enough ε , the ε -shaved ball is worse packing than the unshaved ball. \square

Corollary 3. *For $n = 4$ or 5 , the unit ball B^n is irreducible but not relatively worst packing.*

Of the dimensions where the optimal packing density of the ball is known, the theorems given leave only the cases, $n = 2$ and $n = 3$, where the unique critical lattices, A_2 and D_3 , are critically eutactic, unresolved. We show next that the critical determinant of a nearly spherical body in \mathbb{R}^2 and \mathbb{R}^3 is bounded from above in first order by the average of its support height function (or its radial distance function) evaluated at the minimal vectors of the critical lattice of the ball. The form of the error term depends on whether the support height or the radial distance is used.

Theorem 4. *Suppose that Λ_0 is the unique critical lattice of B^n , Λ_0 is critically eutactic with eutaxy $v_{\mathbf{x}}$ for $\mathbf{x} \in S(\Lambda_0)$.*

- (i) *Let $h_{\mathbf{x}} = 1 + \eta_{\mathbf{x}} = h_K(\mathbf{x})$ be the values of the support height function of a nearly spherical body $(1 - \varepsilon)B \subseteq K \subseteq (1 + \varepsilon)B$ evaluated at $\mathbf{x} \in S(\Lambda_0)$. Then there is a lattice Λ , admissible for K , whose determinant $d(\Lambda)$ satisfies*

$$(4) \quad \frac{d(\Lambda)}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \eta_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\eta_{\mathbf{x}}|^2.$$

- (ii) *Let $r_{\mathbf{x}} = 1 + \rho_{\mathbf{x}} = r_K(\mathbf{x})$ be the values of the radial distance function of a nearly spherical body $(1 - \varepsilon)B \subseteq K \subseteq (1 + \varepsilon)B$ evaluated at $\mathbf{x} \in S(\Lambda_0)$. Then there is a lattice Λ' , admissible for K , whose determinant $d(\Lambda')$ satisfies*

$$\frac{d(\Lambda')}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|,$$

where ε' depends on ε and becomes arbitrarily small as $\varepsilon \rightarrow 0$.

Proof. (i) By Theorem 1 (iv), there exists a symmetric linear map Q such that $\langle \mathbf{x}, Q\mathbf{x} \rangle = \eta_{\mathbf{x}}$ and $\text{trace } Q = \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \eta_{\mathbf{x}}$ and this map depends linearly on the variables $\eta_{\mathbf{x}}$. Consider then the lattice $(\text{Id} + Q)\Lambda$. For all $\mathbf{x} \in S(\Lambda_0)$, the vector $(\text{Id} + Q)\mathbf{x}$ satisfies $\langle (\text{Id} + Q)\mathbf{x}, \mathbf{x} \rangle \geq h_K(\mathbf{x})$ and therefore lies outside the interior of K . Therefore Λ is admissible for K . Because Q depends linearly on $\eta_{\mathbf{x}}$, there is a constant C such that (4) holds.

- (ii) The situation in this case is similar to the situation in the previous case, except we must use the radial distances in the directions $\mathbf{x} \in S(\Lambda_0)$, instead of the support heights in these directions, to construct an admissible lattice. By the construction given in the

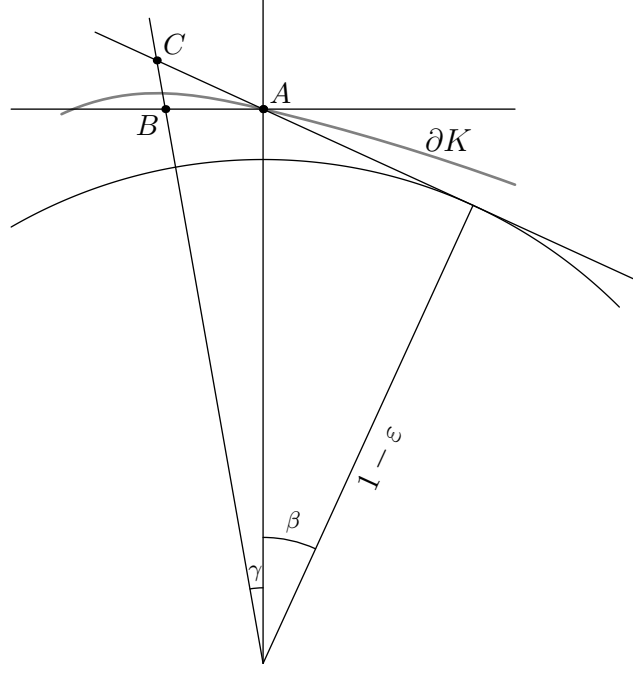


FIGURE 1. Illustration of the construction given in the proof of Thm. 4 to bound the dilation factor needed to ensure that the original point B when dilated to C lies outside the body K .

previous case, we can construct a lattice $\Lambda = T\Lambda_0$, in general not admissible, such that $\langle \mathbf{x}, T\mathbf{x} \rangle = 1 + \rho_{\mathbf{x}}$ and

$$\frac{d(\Lambda)}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|^2.$$

We wish to dilate this lattice by a factor $1 + \alpha$ so that $\Lambda' = (1 + \alpha)\Lambda$ is admissible. Therefore for all $\mathbf{x} \in S(\Lambda_0)$ we must have

$$\alpha \geq \alpha_{\mathbf{x}} = \frac{r_K(T\mathbf{x}/\|T\mathbf{x}\|) - \|T\mathbf{x}\|}{\|T\mathbf{x}\|}.$$

As we only want to use information about the radial function evaluated at $S(\Lambda_0)$ (and the fact that the radial distance function is bounded between $1 - \varepsilon$ and $1 + \varepsilon$), we bound the dilation factor as illustrated in Figure 1. In the plane containing the origin O , $(1 + \rho_{\mathbf{x}})\mathbf{x}$ (denoted A in the figure), and $T\mathbf{x}$ (denoted B), we draw the tangent AX from A to the circle of radius $1 - \varepsilon$ about the origin in the direction away from B . Note that B lies on the line through A perpendicular to OA . Since $\rho_{\mathbf{x}} < \varepsilon$, the angle

$\beta = \widehat{AOX}$ satisfies $\beta \leq \cos^{-1} \frac{1-\varepsilon}{1+\varepsilon} \leq 2\sqrt{\varepsilon}$. By convexity, the continuation of the tangent from A away from the circle must lie outside of K . We mark the intersection of the tangent and the ray OB as C . Then either $\alpha_{\mathbf{x}} \leq 0$, or the boundary of K intersects the ray OB between C and B . Since T depends linearly on $\rho_{\mathbf{x}}$, the angle $\gamma = \widehat{AOB}$ satisfies $\gamma \leq C \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|$ for some constant C . By the law of sines, we have

$$|BC| = \frac{|AB| \sin(\beta)}{\cos(\gamma + \beta)} \leq \frac{(1 + \varepsilon)\gamma\beta}{1 - \frac{1}{2}(\beta + \gamma)^2} \leq (1 - \varepsilon)\varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|,$$

where ε' depends on ε and becomes arbitrarily small as $\varepsilon \rightarrow 0$. Therefore, if we let $\alpha = \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|$, we have that $\alpha \geq \alpha_{\mathbf{x}}$ and $(1 + \alpha)T\mathbf{x}$ is guaranteed to lie outside the interior of K for all $\mathbf{x} \in S(\Lambda_0)$.

The determinant of the dilated lattice $d(\Lambda')$ satisfies

$$\begin{aligned} \frac{d(\Lambda')}{d(\Lambda_0)} &= (1 + \alpha)^n \frac{d(\Lambda)}{d(\Lambda_0)} \\ &\leq (1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|^2) (1 + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|)^n \\ &\leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|, \end{aligned}$$

where the quadratic and higher order terms have been absorbed into the last term. □

4. THE CASE $n = 3$

It is known that the disk B^2 is not relatively worst packing [7]. Therefore, of the dimensions where the densest lattice packing of balls is known, Corollaries 2 and 3 leave only the case $n = 3$ to be resolved. Since in all the other dimensions we have seen that B^n is not relatively worst packing, it may come as somewhat of a surprise, in spite of Ulam's conjecture and the intuitive argument in the introduction, that B^3 , as we will prove in this section, turns out to be relatively worst packing.

The main idea of the proof comes from the fact that, as a consequence of Theorem 4, the critical determinant of a nearly spherical body is, in first order, determined by the sum $\sum_{\mathbf{x} \in S(D_3)} [r(\mathbf{x}) - 1]$ of the change (compared to the sphere) in the radial distance function evaluated at the twelve minimal vectors of D_3 , oriented in such a way as to minimize this sum. The average of this sum over all orientations of the minimal

vectors is proportional to the average change in the radial distance function over the sphere, which is in turn proportional in first order to the change in volume of the body. Therefore, in first order any change to the radial distance function increases the optimal packing density, unless the sum above is independent of the orientation of the minimal vectors, so that the optimal packing density is not affected in first order. It is not hard to show that any even function for which the sum above is independent of the orientation must be (up to a constant term) a second degree spherical harmonic. In first order, such a change to the radial distance function corresponds simply to a linear transformation toward an ellipsoid, which we already know to pack as densely as the sphere. To formalize this argument, we require a few more technical details.

Any even continuous function f on S^2 can be expanded in terms of spherical harmonics $f(\mathbf{x}) = \sum_{l=0, l \text{ even}}^{\infty} f_l(\mathbf{x})$, where $f_l(\mathbf{x})$ is a homogeneous harmonic polynomial of degree l in \mathbf{x} restricted to S^2 , and the series converges at least in $L^2(\sigma)$ (let σ denote the rotation-invariant probability measure on S^2). We may also write $f_l = \pi_l f$, where π_l is the orthogonal projection from $L^2(\sigma)$ to the finite dimensional space of spherical harmonics of degree l .

Lemma 1. *Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that if $K \subset \mathbb{R}^3$ satisfies $(1 - \varepsilon')B^3 \subseteq K \subseteq (1 + \varepsilon')B^3$, then K has a linear image $K' = TK$ that satisfies $(1 - \varepsilon)B^3 \subseteq K' \subseteq (1 + \varepsilon)B^3$ and whose radial function has mean 1 and vanishing second spherical harmonic component.*

Proof. Let Sym_0^3 be the space of zero-trace symmetric linear maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ equipped with Hilbert-Schmidt norm. We identify Sym_0^3 also with the space of zero-trace quadratic forms $\mathbb{R}^3 \rightarrow \mathbb{R}$. Consider the map $F_K : \text{Sym}_0^3 \rightarrow \text{Sym}_0^3$, given by $[F_K(A)](\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle - \pi_2[r_{(1+A)K}](\mathbf{x})$, where A is viewed as a linear map and $F_K(A)$ is viewed as a quadratic form. Note that the space of second degree spherical harmonics is the same as the space of zero-trace quadratic forms.

When K is the unit ball B^3 , we have

$$r_{(1+A)B^3}(\mathbf{x}) = \frac{1}{\|(1+A)^{-1}\mathbf{x}\|} = 1 + \langle \mathbf{x}, A\mathbf{x} \rangle + O(\|A\|^2), \quad (\mathbf{x} \in S^2)$$

When taking the second harmonic component of the above, the constant 1 vanishes, the second term is preserved, and the third term contributes a term that is again of order at most $\|A\|^2$. Therefore, $\|F_{B^3}(A)\| < C\|A\|^2$. Fix a small closed ball $\mathcal{B} \subseteq \text{Sym}_0^3$ around 0 so that $F_{B^3}(\mathcal{B}) \subseteq \frac{1}{2}\mathcal{B}$. Note that $F_K(A)$ can be made arbitrarily close to $F_{B^3}(A)$ uniformly with respect to $A \in \mathcal{B}$ by appropriately choosing

ε' . Then if ε' is small enough, F_K maps \mathcal{B} into \mathcal{B} , and by Brouwer's fixed point theorem there is a fixed point $F_K(A) = A$. Hence, the second spherical harmonic component of $r_{(1+A)K}$ vanishes. The mean of $r_{(1+A)K}$ might not be 1, but all that is left to do is to contract or dilate $(1+A)K$, and this operation does not change the fact that the second spherical harmonic component vanishes. By restricting ε' , the norm of A can be made as small as needed to guarantee that $(1-\varepsilon)B^3 \subseteq K' \subseteq (1+\varepsilon)B^3$. \square

Let \mathbf{p} represent a chosen north pole on S^2 . We say that a function or measure on S^2 is *zonal* if it is invariant under all rotations that preserve \mathbf{p} . If $P_l(t)$ is the Legendre polynomial of degree l , then $h_l(\mathbf{x}) = P_l(\langle \mathbf{x}, \mathbf{p} \rangle)$ is the unique zonal harmonic of degree l whose value at \mathbf{p} is 1. Given a function f and a zonal measure μ on S^2 , we can define their convolution $(\mu * f)(\mathbf{y}) = \int_{S^2} f(\mathbf{x}) d\mu(U_{\mathbf{y}}(\mathbf{x}))$, where $U_{\mathbf{y}}$ is any rotation that maps \mathbf{y} to \mathbf{p} . Convolution with a zonal measure acts as a multiplier transformation on the harmonic expansion, i.e., $(\mu * f)(\mathbf{x}) = \sum_{l=0}^{\infty} c_l f_l(\mathbf{x})$. The multiplier coefficients can be found by performing the convolution on the zonal harmonic $c_l = (\mu * h_l)(\mathbf{p}) = \int_{S^2} h_l(\mathbf{x}) d\mu(\mathbf{x})$ [11].

We now go on to prove two lemmas related to the configuration of minimal vectors in D_3 . Let D_3 be rotated so that \mathbf{p} is one of the minimal vectors. Then there is a unique zonal measure μ such that for every continuous zonal function f ,

$$\int_{S^2} f(\mathbf{y}) d\mu(\mathbf{y}) = \frac{1}{2} \sum_{\mathbf{x} \in S(D_3)} f(\mathbf{x}).$$

Since $\langle \mathbf{x}, \mathbf{p} \rangle = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$ with multiplicities 1, 4, 2, 4, 1 as \mathbf{x} ranges over $S(D_3)$, we have that μ is the zonal measure with total weight 6, distributed as follows: weight 1/2 on each of the north and south poles, weight 2 on each line of latitude 30° north and south, and weight 1 on the equator.

Lemma 2. *Let $c_l = P_l(1) + 4P_l(\frac{1}{2}) + P_l(0) = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. Then $c_l = 0$ if and only if $l = 2$. Moreover, $|c_l - 1| < Cl^{-1/2}$ for some constant C .*

Proof. It would here be advantageous to work with the rescaled Legendre polynomials $Q_l(t) = 2^l l! P_l(t)$. Their recurrence relation is given by $Q_{l+1}(t) = 2(2l+1)tQ_l(t) - 4l^2 Q_{l-1}(t)$. It is clear from the recurrence relation and the base cases $Q_0(t) = 1$ and $Q_1(t) = 2t$ that the values of $Q_l(t)$ at $t = 0, \frac{1}{2}, 1$ are integers. We are interested in their residues modulo 8. For $t = 0$, $Q_{l+2}(0)/2^{l+2} = -(l+1)^2 Q_l(0)/2^l$,

and so by induction $Q_l(0)$ is divisible by 2^l . For $t = \frac{1}{2}$, $Q_{l+1}(\frac{1}{2})$ is odd whenever $Q_l(\frac{1}{2})$ is odd, and hence by induction for all l . For $t = 1$, $Q_l(1) = 2^l l! P_l(1) = 2^l l!$ is also divisible by 2^l . Therefore, for all $l \geq 3$ we have $2^l l! c_l = Q_l(0) + 4Q_l(\frac{1}{2}) + Q_l(1) \equiv 4 \pmod{8}$ and c_l cannot vanish. For $l = 2$, we have $P_l(t) = \frac{3}{2}t^2 - \frac{1}{2}$ and $c_l = 1 + 4 \cdot (-\frac{1}{8}) - \frac{1}{2} = 0$. The second part of the lemma follows from the bound $|P_l(t)| < (\pi l \sqrt{1-t^2}/2)^{-1/2}$ [12]. \square

Lemma 3. *Let μ be the zonal measure described above and let Φ be the operator of convolution with μ . Then its multiplier coefficients are $c_l = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. Further, let Z be the space, equipped with the $L^1(\sigma)$ norm, of even functions f on S^2 for which $f_2 = 0$. Then Φ maps Z to Z , and as an operator $Z \rightarrow Z$ it is one-to-one, bounded, and has a bounded inverse.*

Proof. As noted above, the multiplier coefficients of a convolution are given by $c_l = \int_{S^2} P_l(\langle \mathbf{x}, \mathbf{p} \rangle) d\mu(\mathbf{x}) = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. As a consequence of Lemma 2, Φ clearly maps Z to Z and is one-to-one on Z . Since Φ is a convolution operator with a finite measure, it is clearly a bounded operator $Z \rightarrow Z$.

We now construct an operator $\Psi : Z \rightarrow Z$, which we then show to be the inverse of Φ . Let

$$\Psi = \sum_{k=0}^3 (\text{Id} - \Phi)^k + \sum_{l \text{ even}, l \neq 2} c_l^{-1} (1 - c_l)^4 \pi_l.$$

The norm of π_l induced by the L^1 norm does not exceed the L^1 norm of the projection kernel $K_l(\mathbf{x}, \cdot)$, but $\|K_l(\mathbf{x}, \cdot)\|_1 \leq \|K_l(\mathbf{x}, \cdot)\|_2 = \sqrt{2l+1}$. Therefore, the sum $\sum_{l \text{ even}, l \neq 2} |c_l^{-1}| \cdot |1 - c_l|^4 \cdot \|\pi_l\|$ converges and so Ψ is bounded. Note also that Ψ is a multiplier transform and its multiplier coefficients are simply c_l^{-1} for all even $l \neq 2$. Therefore $\Psi = \Phi^{-1}$. \square

We can now prove the main result.

Theorem 5. *There exists $\varepsilon > 0$ such that if $K \subset \mathbb{R}^3$ is a non-ellipsoidal origin-symmetric convex solid and $(1 - \varepsilon)B^3 \subseteq K \subseteq (1 + \varepsilon)B^3$, then $\varphi(K) > \varphi(B^3)$. In other words, B^3 is relatively worst packing.*

Proof. Given Lemma 1 and the fact that $\varphi(K)$ is invariant under linear transformations of K , we may assume without loss of generality that K is a non-spherical solid whose radial function has an expansion in spherical harmonics of the form

$$r_K(\mathbf{x}) = 1 + \rho(\mathbf{x}) = 1 + \sum_{l \text{ even}, l \geq 4} \rho_l(\mathbf{x}).$$

Therefore, the volume of K satisfies

$$\text{vol } K = \frac{4\pi}{3} \int_{S^2} r_K^3 d\sigma \geq \frac{4\pi}{3} \left(\int_{S^2} r_K d\sigma \right)^3 = \frac{4\pi}{3}.$$

We consider all the rotations $U(K)$ of the solid K and the determinants of the admissible lattices obtained when the construction of Theorem 4 (ii) is applied to $U(K)$. Note that the determinants obtained depend only on $\rho(U^{-1}(\mathbf{x}))$ for $\mathbf{x} \in S(D_3)$. Let us define $\Delta_K = \frac{\phi(K)^{-1}}{\phi(B^n)^{-1}} - 1$. Combining our bound on $\text{vol } K$ with Theorem 4 (ii) we get

$$(5) \quad \Delta_K \leq \min_{U \in SO(3)} \left[\frac{1}{4} \sum_{\mathbf{x} \in S(D_3)} \rho(U^{-1}(\mathbf{x})) + \varepsilon' \sum_{\mathbf{x} \in S(D_3)} |\rho(U^{-1}(\mathbf{x}))| \right].$$

We may pick a single point $\mathbf{x}_0 \in S(D_3)$ and decompose $SO(3)$ into subsets $\mathcal{U}_{\mathbf{y}}$ of all rotations such that $U^{-1}(\mathbf{x}_0) = \mathbf{y}$. In each subset $\mathcal{U}_{\mathbf{y}}$ the minimum on the right hand side of (5) is no larger than the average value over $\mathcal{U}_{\mathbf{y}}$ (with respect to the obvious uniform measure). This averaging procedure transforms (5) into

$$(6) \quad \Delta_K \leq \min_{\mathbf{y} \in S^2} \left[\frac{1}{4} \Phi[\rho](\mathbf{y}) + \varepsilon' \Phi[|\rho|](\mathbf{y}) \right],$$

where Φ is the same convolution operator as in Lemma 3.

For integrable functions f and g over a domain of measure 1, such that f is of zero mean and g is non-negative, we have $\|\min(f, 0)\|_1 \leq \|g\|_1 - \min[f + g]$. Since $\|f\|_1 = 2\|\min(f, 0)\|_1$, we have $\min[f + g] \leq -\frac{1}{2}\|f\|_1 + \|g\|_1$. Therefore, if we let $f = \frac{1}{4}\Phi[\rho](\mathbf{y})$ and let $g = \varepsilon'\Phi[|\rho|](\mathbf{y})$, we obtain

$$\Delta_K \leq -\frac{1}{8}\|\Phi^{-1}\|^{-1} \cdot \|\rho\|_1 + 2\varepsilon'\|\Phi\| \cdot \|\rho\|_1.$$

Since ε' can be made as small as needed by decreasing ε we conclude that there is a coefficient $c > 0$ such that $\Delta_K \leq -c\|\rho\|_1$. \square

Remark 1. *Since the optimal lattice packing density of B^3 is also its optimal packing density for non-lattice packings [6], it follows from the theorem that any sufficiently spherical, origin-symmetric solid also packs better than the ball when packing is taken to allow rotations and non-lattice translations. The case of origin-non-symmetric solid remains open.*

Remark 2. *Note that the theorem provides a bound for the improvement in the packing density of K over B^3 which is linear in terms of the L^1 norm of the difference between the radial functions of B^3 and an appropriate linear transformation of K .*

Remark 3. *It appears that the size of the neighborhood ε that can be obtained using this method of proof is primarily governed by $\|\Phi^{-1}\|^{-1}$, which is in turn primarily governed by $\min_{l \geq 4} |c_l|$. By numerical computation, we see that $|c_l| > 0.2$ for all even $l \geq 4$, except for $l = 10$, for which $c_l \approx 10^{-3}$.*

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